

JONQUIÈRES MAPS AND $SL_2(\mathbb{C})$ -COCYCLES

by

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Abstract. — We start the study of the family of birational maps $(f_{\alpha,\beta})$ of $\mathbb{P}_{\mathbb{C}}^2$ in [11]. For generic α and β of modulus 1 the centraliser of $f_{\alpha,\beta}$ is trivial, the topological entropy of $f_{\alpha,\beta}$ is 0, there exist two areas of linearisation: in the first one the closure of the orbit of a point is a torus, in the other one the closure of the orbit of a point is the union of two circles. What does happen between these two areas ? On $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ any $f_{\alpha,\beta}$ can be viewed as a cocycle (β, A_{α}) ; using recent results about $SL_2(\mathbb{C})$ -cocycles ([1]) we can determine the LYAPUNOV exponent of the cocycle associated to $f_{\alpha,\beta}$.

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1. Introduction

In this article we deal with a family of birational maps $(f_{\alpha,\beta})$ given by

$$f_{\alpha,\beta}: \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2 \quad (x:y:z) \dashrightarrow ((\alpha x + y)z : \beta y(x+z) : z(x+z))$$

where α, β denote two complex numbers with modulus 1, case where we know almost nothing about the dynamics. This family of maps satisfies the following properties ([11]):

- for α and β generic the centraliser

$$C(f_{\alpha,\beta}) = \{g \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2) \mid g \circ f_{\alpha,\beta} = f_{\alpha,\beta} \circ g\}$$

- of $f_{\alpha,\beta}$ is isomorphic to \mathbb{Z} ;
- the topological entropy of $f_{\alpha,\beta}$ is 0;
- there exists a neighborhood \mathcal{U} of $(0:0:1)$ such that for any $q \in \mathcal{U}$ the closure of the orbit of q under $f_{\alpha,\beta}$ is a torus;
- there exists a neighborhood \mathcal{V} of $(0:1:0)$ such that for any $q \in \mathcal{V}$ the closure of the orbit of q under $f_{\alpha,\beta}^2$ is a circle.

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We can also see $f_{\alpha,\beta}$ on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ (since all the computations of [11] have been done in an affine chart they may all be carried on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$); the sets $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{S}_{\rho}^1$, where $\mathbb{S}_{\rho}^1 = \{y \in \mathbb{C} \mid |y| = \rho\}$, are invariant.

Let us define $A_n^{\alpha,\rho} : \mathbb{S}_{\rho}^1 \rightarrow M_2(\mathbb{C})$ given in terms of $A^{\alpha,\rho}(y) = \begin{bmatrix} \alpha & y \\ 1 & 1 \end{bmatrix}$ by

$$A_n^{\alpha,\rho}(\cdot) = A^{\alpha,\rho}(\beta^n \cdot) A^{\alpha,\rho}(\beta^{n-1} \cdot) \dots A^{\alpha,\rho}(\beta \cdot) A^{\alpha,\rho}(\cdot).$$

To compute $f_{\alpha,\beta}^n(x, y)$ is equivalent to compute $A_n^{\alpha,\rho}(y)$ as soon as $f_{\alpha,\beta}^k(x, y) \neq (-1, \alpha)$ for any $1 \leq k \leq n$.

Using [1] we are able to determine the LYAPUNOV exponent of the cocycle $(\beta, A^{\alpha,\rho})$.

Theorem A. — *The LYAPUNOV exponent of $(\beta, A^{\alpha,\rho})$ is*

- positive as soon as $\rho > 1$;
- zero as soon as $\rho \leq 1$.

More precisely $f_{\alpha,\beta}$ is semi conjugate to $\left(\frac{\alpha x + y^2}{x+1}, \beta^{1/2} y\right)$ and the LYAPUNOV exponent of the cocycle $(\beta^{1/2}, B^{\alpha,\rho})$, where

$$B^{\alpha,\rho}(y) = \begin{bmatrix} \alpha & y^2 \\ 1 & 1 \end{bmatrix},$$

is equal to $\max(0, \ln \rho)$.

Organisation of the article. — We introduce the family $f_{\alpha,\beta}$ and its properties (§2). Then in §3 we deal with the recent works of AVILA on $SL_2(\mathbb{C})$ -cocycles that we need to establish the proof of Theorem A (see §4). Let us explain the sketch of the proof of Theorem A. The cocycles $(\beta, A^{\alpha,\rho})$ and $(\beta^{1/2}, B^{\alpha,\rho})$ are conjugate; we associate to $(\beta^{1/2}, B^{\alpha,\rho})$ a cocycle $(\beta^{1/2}, \tilde{B}^{\alpha,\rho})$ that belongs to $SL_2(\mathbb{C})$. We first determine $\lim_{\rho \rightarrow 0} L(\beta^{1/2}, \tilde{B}^{\alpha,\rho})$ and then look at $\lim_{\rho \rightarrow +\infty} L(\beta^{1/2}, \tilde{B}^{\alpha,\rho})$; in both cases, we get 0. Using [1, Theorem 5] we obtain that $L(\beta^{1/2}, \tilde{B}^{\alpha,\rho})$ vanishes everywhere; it allows us to determine $L(\beta, A^{\alpha,\rho})$ since

$$L(\beta^{1/2}, B^{\alpha,\rho}(y)) = L(\beta^{1/2}, \tilde{B}^{\alpha,\rho}(y)) + \max(0, \ln \rho).$$

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2. Some properties of the family $f_{\alpha,\beta}$

A birational map ϕ from $\mathbb{P}_{\mathbb{C}}^2$ into itself is a map of the form

$$(x : y : z) \mapsto (\phi_0(x, y, z) : \phi_1(x, y, z) : \phi_2(x, y, z)),$$

where the ϕ_i 's are some homogeneous polynomials of the same degree without common factor, which admits an inverse of the same type. The *degree* of ϕ , denoted $\deg \phi$, is the degree of the ϕ_i 's.

The degree is not a birational invariant: $\deg \psi\phi\psi^{-1} \neq \deg \phi$ for generic birational maps ϕ and ψ . The *first dynamical degree* of ϕ given by

$$\lambda(\phi) = \lim_{n \rightarrow +\infty} (\deg \phi^n)^{1/n},$$

is a birational invariant; it is strongly related to the topological entropy $h_{\mathrm{top}}(\phi)$ of ϕ (see [16, 19])

$$h_{\mathrm{top}}(\phi) \leq \log \lambda(\phi) \quad (2.1)$$

Any birational map ϕ admits a resolution

$$\begin{array}{ccc} & S & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}_{\mathbb{C}}^2 & \dashrightarrow \phi \dashrightarrow & \mathbb{P}_{\mathbb{C}}^2 \end{array}$$

where $\pi_1, \pi_2: S \rightarrow \mathbb{P}_{\mathbb{C}}^2$ are sequences of blow-ups (see [3] for example). The resolution is *minimal* if and only if no (-1) -curve of S is contracted by both π_1 and π_2 . The *base-points* of ϕ are the points blown-up in π_1 , which can be points of $\mathbb{P}_{\mathbb{C}}^2$ or infinitely near points. We denote by $\mathfrak{b}(\phi)$ the number of such points, which is also equal to the difference of the ranks of $\mathrm{Pic}(S)$ and $\mathrm{Pic}(\mathbb{P}_{\mathbb{C}}^2)$, and thus equals to $\mathfrak{b}(\phi^{-1})$. We denote by $\mathfrak{b}(\phi)$ the number of base-points of ϕ . The *dynamical number of base-points* of ϕ introduced in [9] is by definition

$$\mu(\phi) = \lim_{n \rightarrow +\infty} \frac{\mathfrak{b}(\phi^n)}{n}$$

it is a real positive number that satisfies $\mu(\phi^n) = |n\mu(\phi)|$ for any $n \in \mathbb{Z}$; it also allows to give a characterization of birational maps conjugate to automorphisms:

Theorem 2.1 ([9]). — *Let S be a smooth projective surface; the birational map $\phi \in \mathrm{Bir}(S)$ is conjugate to an automorphism of a smooth projective surface if and only if $\mu(\phi) = 0$.*

The behavior of a birational map ϕ from $\mathbb{P}_{\mathbb{C}}^2$ into itself is strongly related to the behavior of $(\deg \phi^n)_n$ (see [15, 14, 9]); up to birational conjugacy exactly one of the following holds:

- the sequence $(\deg \phi^n)_n$ is bounded and either ϕ is of finite order, or ϕ is an automorphism of $\mathbb{P}_{\mathbb{C}}^2$;
- there exists an integer k such that

$$\lim_{n \rightarrow +\infty} \frac{\deg \phi^n}{n} = k^2 \frac{\mu(\phi)}{2}$$

and ϕ is not an automorphism;

- there exists an integer $k \geq 3$ such that

$$\lim_{n \rightarrow +\infty} \frac{\deg \phi^n}{n^2} = k^2 \frac{\kappa(\phi)}{9}$$

where $\kappa(\phi) \in \mathbb{Q}$ is a birational invariant and ϕ is an automorphism;

- the sequence $(\deg \phi^n)_n$ grows exponentially (see [14] for more precise dynamical properties).

In the first three cases $\lambda(\phi) = 1$, in the last one $\lambda(\phi) > 1$. In case 2. (resp. 3.) the map ϕ preserves a unique fibration which is rational (resp. elliptic).

Definition. — In case 1. (resp. 2., resp. 3, resp. 4) we say that ϕ is *elliptic* (resp. a JONQUIÈRES twist, resp. an HALPHEN twist, resp. *hyperbolic*).

Let us give some examples. Let

$$\phi(x, y) = \left(\frac{a(y)x + b(y)}{c(y)x + d(y)}, \frac{\alpha y + \beta}{\gamma y + \delta} \right)$$

with

$$\begin{bmatrix} a(y) & b(y) \\ c(y) & d(y) \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{C}(y)), \quad \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in \mathrm{PGL}_2(\mathbb{C})$$

be an element of the JONQUIÈRES group $\mathrm{PGL}_2(\mathbb{C}(y)) \rtimes \mathrm{PGL}_2(\mathbb{C})$; either ϕ is elliptic (for instance $\phi: (x:y:z) \dashrightarrow (yz:xz:xy)$) or ϕ is a JONQUIÈRES twist (for example $\phi: (x:y:z) \dashrightarrow (xz:xy:z^2)$ for which the unique invariant fibration is $y/z = \text{constant}$). Consider the family

$$\phi_\varepsilon: \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2 \quad (x:y:z) \dashrightarrow ((y+z-\varepsilon z)(y+z):x(y-\varepsilon z):z(y+z));$$

if ε is $1/2$ or $1/3$ then ϕ_ε is an HALPHEN twist ([14, Proposition 9.5]). HÉNON automorphisms give by homogeneization examples of hyperbolic maps.

Clearly elliptic birational maps have a poor dynamical behavior. The study of automorphisms of positive entropy is strongly related with birational maps of the complex projective plane:

Theorem 2.2 ([10]). — *Let S be a compact complex surface that carries an automorphism f of positive topological entropy.*

- *either the KODAIRA dimension of S is zero and f is conjugate to an automorphism on the unique minimal model of S that necessarily is a torus, or a K3 surface or an ENRIQUES surface;*
- *or the surface S is a non-minimal rational one, isomorphic to $\mathbb{P}_{\mathbb{C}}^2$ blown up at n points, $n \geq 10$, and f is conjugate to a birational map of $\mathbb{P}_{\mathbb{C}}^2$.*

This yields a lot of examples of hyperbolic birational maps for which we can establish dynamical properties ([17, 4, 5, 6, 7, 13, 12]). In this article we consider the JONQUIÈRES maps

$$f_{\alpha,\beta}: \mathbb{P}_{\mathbb{C}}^2 \dashrightarrow \mathbb{P}_{\mathbb{C}}^2 \quad (x:y:z) \dashrightarrow ((\alpha x + y)z: \beta y(x+z): z(x+z))$$

where α, β denote two complex numbers with modulus 1. The base-points of $f_{\alpha,\beta}$ are

$$(1:0:0), \quad (0:1:0), \quad (-1:\alpha:1).$$

Any $f_{\alpha,\beta}$ preserves a rational fibration (the fibration $y = \text{constant}$ in the affine chart $z = 1$). Each element of the family $(f_{\alpha,\beta})$ has first dynamical degree 1 hence topological entropy zero (see (2.1)); more precisely one has ([9, Example 4.3], [11, Lemma 1.4])

$$\mu(f_{\alpha,\beta}) = \frac{1}{2}$$

so $f_{\alpha,\beta}$ is not conjugate to an automorphism (Theorem 2.1). A way to measure the chaos is to look at the size of centralizers; the centralizer of $f_{\alpha,\beta}$ is isomorphic to \mathbb{Z} (see [11, Theorem 1.6]).

The idea of the proof is the following: the point $p = (1 : \alpha : 1)$ is blown-down onto a fiber of the fibration $y = \text{constant}$. Let ψ be an element of

$$C(f_{\alpha,\beta}) = \{g \in \text{Bir}(\mathbb{P}_{\mathbb{C}}^2) \mid g \circ f_{\alpha,\beta} = f_{\alpha,\beta} \circ g\}$$

since ψ blows down a finite number of curves there exists a positive integer k (chosen minimal) such that $f_{\alpha,\beta}^k(p)$ is not blown down by ψ . Replacing ψ by $\tilde{\psi} = \psi f_{\alpha,\beta}^{k-1}$ one gets that $\tilde{\psi}(p)$ is an indeterminacy point of $f_{\alpha,\beta}$. In other words $\tilde{\psi}$ permutes the indeterminacy points of $f_{\alpha,\beta}$. A more precise study allows us to establish that p is fixed by $\tilde{\psi}$. The parameters α, β being generic, the closure of the negative orbit of p under the action of $f_{\alpha,\beta}$ is ZARISKI dense; since $\tilde{\psi}$ fixes any element of the orbit of p one obtains $\tilde{\psi} = \text{id}$.

We can also see $f_{\alpha,\beta}$ on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$ and that is what we will do in the sequel (since all the computations of [11] have been done in an affine chart they may all be carried on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{P}_{\mathbb{C}}^1$); the sets $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{S}_{\mathbb{P}}^1$ are invariant. In [11] we show that $f_{\alpha,\beta}^2$ has two linearizable fixed points for generic α, β ; near these points, the closure of all orbits are circles or tori (in this second case we give here a more precise statement than in [11]).

Theorem 2.3 ([11]). — *If α and β are generic, there exists a strictly positive real number r such that $f_{\alpha,\beta}$ is conjugate to $(\alpha x, \beta y)$ on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{D}(0, r)$ where $\mathbb{D}(0, r)$ denotes the disk centered at the origin with radius r .*

Remark 2.4. — The point $(\alpha - 1, 0)$ is also a fixed point of $f_{\alpha,\beta}$ where the behavior of $f_{\alpha,\beta}$ is the same as in Theorem 2.3.

Theorem 2.5. — *If α and β are generic, there exists a strictly positive real number r such that $\left(\frac{1}{x}, \frac{1}{y}\right) f_{\alpha,\beta}^2 \left(\frac{1}{x}, \frac{1}{y}\right)$ is conjugate to $\left(\frac{x}{\beta}, \frac{y}{\beta^2}\right)$ on $\mathbb{P}_{\mathbb{C}}^1 \times \mathbb{D}(0, r)$ where $\mathbb{D}(0, r)$ denotes the disk centered at the origin with radius r .*

Proof. — Let us consider the map $\psi(x, y) = \left(\frac{a(y)x+b(y)}{c(y)x+1}, y\right)$. The equation

$$\psi^{-1} \left(\frac{1}{x}, \frac{1}{y} \right) f_{\alpha,\beta}^2 \left(\frac{1}{x}, \frac{1}{y} \right) \psi = \left(\frac{x}{\beta}, \frac{y}{\beta^2} \right)$$

yields to

$$\begin{aligned} & \beta a(\beta^{-2}y)c(y) + \beta a(\beta^{-2}y)a(y) - c(\beta^{-2}y)a(y) + \alpha a(\beta^{-2}y)a(y) \\ & + y(\alpha^2 a(\beta^{-2}y)c(y) - \alpha c(\beta^{-2}y)c(y) - c(\beta^{-2}y)c(y) - c(\beta^{-2}y)a(y)) = 0, \end{aligned} \quad (2.2)$$

$$\begin{aligned} & \beta a(\beta^{-2}y) - \beta a(y) + y(\alpha^2 a(\beta^{-2}y) - \alpha \beta c(y) - \beta c(y) - \beta a(y) - \alpha c(\beta^{-2}y) - c(\beta^{-2}y)) \\ & + \beta(\alpha + \beta)a(y)b(\beta^{-2}y) + (\alpha + \beta)b(y)a(\beta^{-2}y) + \beta^2 b(\beta^{-2}y)c(y) - b(y)c(\beta^{-2}y) \\ & + y(\alpha^2 \beta b(\beta^{-2}y)c(y) - b(y)c(\beta^{-2}y)) = 0 \end{aligned} \quad (2.3)$$

and

$$(\alpha + 1)y + b(y) - \beta b(\beta^{-2}y) - \alpha^2 yb(\beta^{-2}y) + yb(y) - (\alpha + \beta)b(\beta^{-2}y)b(y) = 0 \quad (2.4)$$

Let us set

$$a(y) = \sum_{i \geq 0} a_i y^i, \quad b(y) = \sum_{i \geq 0} b_i y^i, \quad c(y) = \sum_{i \geq 0} c_i y^i.$$

We easily get $a_0 = 1 - \beta$, $b_0 = 0$ and $c_0 = \alpha + \beta$.

The relation (2.4) implies that

$$b_1 = \frac{\beta(1+\alpha)}{1-\beta} \quad \& \quad \beta b_v (1 - \beta^{1-2v}) + F_i(b_i | i < v) = 0 \quad \forall v > 1.$$

Equality (2.3) yields to

$$a_v (\beta^{1-2v} - \beta) + b_v \left((\alpha + \beta) a_0 (1 + \beta^{1-2v}) + c_0 (\beta^{2-2v} - 1) \right) + F_i(a_i, b_i, c_i | i < v) = 0$$

and (2.2) to

$$c_v a_0 (\beta - \beta^{-2v}) + a_v \left((\alpha + \beta) a_0 (1 + \beta^{-2v}) + c_0 (\beta^{1-2v} - 1) \right) + F_i(a_i, b_i, c_i | i < v) = 0$$

where the F_i 's denote universal polynomials; this allows to compute b_v , a_v and c_v . Thus we get a formal conjugacy. According to SIEGEL's theorem ([18]) any linearizing map is convergent on a polydisc; so $a(y)$, $b(y)$ and $c(y)$ are convergent which gives the result. \square

3. Study of $\mathrm{SL}_2(\mathbb{C})$ -cocycles

Let us consider one-dimensional SCHRÖDINGER operators with an analytic one-frequency potential that is

$$H = H_{\beta, \nu}: \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z})$$

given by

$$(Hu)_n = u_{n+1} + u_{n-1} + \nu(n\beta)u_n$$

where $\nu: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{R}$ is an analytic function, called *the potential*, and $\beta \in \mathbb{R} \setminus \mathbb{Q}$, called *the frequency*. Denote by $\Sigma = \Sigma_{\beta, \nu}$ the spectrum of H . For any energy E in \mathbb{R} let us define

$$A(y) = A^{(E-\nu)}(y) = \begin{bmatrix} E - \nu(y) & -1 \\ 1 & 0 \end{bmatrix}, \quad A_n(y) = A(y + (n-1)\beta) \dots A(y) \quad (3.1)$$

which are analytic functions with values in $\mathrm{SL}_2(\mathbb{R})$. They are relevant to the analysis of H because a formal solution of $Hu = Eu$ satisfies

$$\begin{bmatrix} u_n \\ u_{n-1} \end{bmatrix} = A_n(0) \begin{bmatrix} u_0 \\ u_{-1} \end{bmatrix}.$$

The *Lyapunov exponent* at energy E is denoted by $L(E)$ and given by

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}/\mathbb{Z}} \ln \|A_n(y)\| dy \geq 0. \quad (3.2)$$

Energies $E \in \Sigma$ can be

- *supercritical* if $L(E) > 0$;
- *subcritical* if there is a uniform subexponential bound $\ln \|A_n(y)\| = o(n)$ through some band $|\mathrm{Im} y| < \varepsilon$;
- *critical* otherwise.

Progress have been made mainly into the understanding of the behavior of supercritical and subcritical energies; but until [1, 2] there was no global theory of such operators and the transition between supercritical and subcritical energies was not understood. In general subcritical and supercritical regimes can coexist in the spectrum of the same operator ([8]). However it may not be necessary to pass through the critical regime to go from the subcritical and supercritical ones.

In the dynamical approach the understanding of the SCHRÖDINGER operator is obtained through the detailed description of a certain family of dynamical systems. A (one-frequency, analytic) *quasiperiodic* $SL_2(\mathbb{C})$ -cocycle is a pair (β, A) , where $\beta \in \mathbb{R}$ and

$$A: \mathbb{R}/\mathbb{Z} \rightarrow SL_2(\mathbb{C})$$

is analytic, understood as defining a linear skew product acting on $\mathbb{R}/\mathbb{Z} \times \mathbb{C}^2$ by

$$(y, x) \mapsto (y + \beta, A(y) \cdot x).$$

The iterates of the cocycles are given by $(n\beta, A_n)$ where A_n is given by (3.1). The LYAPUNOV exponent $L(\beta, A)$ of the cocycle (β, A) is given by the left hand side of (3.2). We say that (β, A) is *uniformly hyperbolic* if there exist analytic functions

$$u, s: \mathbb{R}/\mathbb{Z} \rightarrow \mathbb{P}_{\mathbb{C}}^2,$$

called the *unstable and stable directions*, and $n \geq 1$ such that for every $y \in \mathbb{R}/\mathbb{Z}$,

$$A(y) \cdot u(y) = u(y + \beta) \quad \text{and} \quad A(y) \cdot s(y) = s(y + \beta),$$

and for every unit vector $x \in s(y)$ (resp. $x \in u(y)$) we have $\|A_n(y) \cdot x\| < 1$ (resp. $\|A_n(y) \cdot x\| > 1$). The unstable and stable directions are uniquely characterized by those properties, and clearly $u(y) \neq s(y)$ for every $y \in \mathbb{R}/\mathbb{Z}$. If (β, A) is uniformly hyperbolic then $L(\beta, A) > 0$. Let us denote by

$$\mathcal{UH} \subset C^0(\mathbb{R}/\mathbb{Z}, SL_2(\mathbb{C}))$$

the set of A such that (β, A) is uniformly hyperbolic. Uniform hyperbolicity is a stable property: \mathcal{UH} is open and $A \mapsto L(\beta, A)$ is analytic over \mathcal{UH} : regularity properties of the LYAPUNOV exponent are consequence of the regularity of the unstable and stable directions which depend smoothly on both variables. If $L(\beta, A) > 0$ but $(\beta, A) \notin \mathcal{UH}$ then we say that (β, A) is *nonuniformly hyperbolic* and denote it by \mathcal{NH} .

Most important examples are SCHRÖDINGER cocycles and $L(E) = L(\beta, A^{(E-\nu)})$. One of the most basic aspects of the connection between spectral and dynamical properties is that $E \notin \Sigma_{\beta, \nu}$ if and only if $(\beta, A^{(E-\nu)})$ is \mathcal{UH} .

If $A \in C^0(\mathbb{R}/\mathbb{Z}, SL_2(\mathbb{C}))$ admits a holomorphic extension to $|\text{Im } y| < \delta$ then for $|\varepsilon| < \delta$ we can define $A_\varepsilon \in C^0(\mathbb{R}/\mathbb{Z}, SL_2(\mathbb{C}))$ by

$$A_\varepsilon(y) = A(y + i\varepsilon).$$

The LYAPUNOV exponent $L(\beta, A_\varepsilon)$ is a convex function of ε . We can thus introduce the function *acceleration* given by

$$\omega(\beta, A) = \lim_{\varepsilon \rightarrow 0^+} \frac{1}{2\pi\varepsilon} (L(\beta, A_\varepsilon) - L(\beta, A)).$$

Since the LYAPUNOV exponent is a convex and continuous function the acceleration is an upper semi-continuous function in $\mathbb{R} \setminus \mathbb{Q} \times C^0(\mathbb{R}/\mathbb{Z}, SL_2(\mathbb{C}))$. The acceleration is quantized:

Theorem 3.1 ([1]). — *If (β, A) is a $\mathrm{SL}_2(\mathbb{C})$ -cocycle with $\beta \in \mathbb{R} \setminus \mathbb{Q}$, then $\omega(\beta, A)$ is always an integer.*

A direct consequence is the following:

Corollary 3.2. — *The function $\varepsilon \mapsto L(\beta, A_\varepsilon)$ is a piecewise affine function of ε .*

It is thus natural to introduce the notion of regularity: $(\beta, A) \in \mathbb{R} \setminus \mathbb{Q} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}_2(\mathbb{C}))$ is regular if $L(\beta, A_\varepsilon)$ is affine for ε in a neighborhood of 0. In other words (β, A) is regular if the equality

$$L(\beta, A_\varepsilon) - L(\beta, A) = 2\pi\varepsilon\omega(\beta, A)$$

holds for all ε small, and not only for the positive ones. Regularity is equivalent to the acceleration being locally constant near (β, A) . It is an open condition in $\mathbb{R} \setminus \mathbb{Q} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}_2(\mathbb{C}))$. The following statement gives a characterization of the dynamics of regular cocycles with positive LYAPUNOV exponent:

Theorem 3.3 ([1]). — *Let (β, A) be a $\mathrm{SL}_2(\mathbb{C})$ -cocycle with $\beta \in \mathbb{R} \setminus \mathbb{Q}$. Assume that $L(\beta, A) > 0$; then (β, A) is regular if and only if (β, A) is \mathcal{UH} .*

One striking consequence is the following:

Corollary 3.4 ([1]). — *For any (β, A) in $\mathbb{R} \setminus \mathbb{Q} \times C^\omega(\mathbb{R}/\mathbb{Z}, \mathrm{SL}_2(\mathbb{C}))$ there exists ε_0 such that*

- $L(\beta, A_\varepsilon) = 0$ (and $\omega(\beta, A) = 0$) for every $0 < \varepsilon < \varepsilon_0$,
- or $(\beta, A_\varepsilon) \in \mathcal{UH}$ for every $0 < \varepsilon < \varepsilon_0$.

4. Proof of Theorem A

Suppose that $\rho \neq 1$. Let us consider the cocycle $(\beta^{1/2}, B^{\alpha, \rho})$ where

$$B^{\alpha, \rho}(y) = \begin{bmatrix} \alpha & y^2 \\ 1 & 1 \end{bmatrix}.$$

Since

$$\left(\frac{\alpha x + y}{x + 1}, \beta y \right) (x, y^2) = (x, y^2) \left(\frac{\alpha x + y^2}{x + 1}, \beta^{1/2} y \right)$$

the cocycles $(\beta, A^{\alpha, \rho})$ and $(\beta^{1/2}, B^{\alpha, \rho})$ have the same behavior. Using two different arguments of monodromy (one for $\rho < 1$ and the other one for $\rho > 1$) we see that there is a continuous determination for the square root of $\det B^{\alpha, \rho}(y) = \alpha - y^2$. Let us thus set

$$\tilde{B}^{\alpha, \rho}(y) = \frac{1}{\sqrt{\alpha - y^2}} B^{\alpha, \rho}(y) \in \mathrm{SL}_2(\mathbb{C}).$$

We first study $L(\beta^{1/2}, \tilde{B}^{\alpha, \rho})$ when ρ is near 0. We note that $\tilde{B}^{\alpha, \rho}$ is almost constant when ρ is near 0; hence $\tilde{B}^{\alpha, \rho}$ is regular with $\omega(\beta^{1/2}, \tilde{B}^{\alpha, \rho}) = 0$ so $L(\beta^{1/2}, \tilde{B}^{\alpha, \rho})$ does not depend on ρ and $L(\beta^{1/2}, \tilde{B}^{\alpha, \rho}) = 0$ when ρ is close to 0.

Let us now determine the behavior of $L(\beta^{1/2}, \tilde{B}^{\alpha, \rho})$ when ρ is near $+\infty$. Remark that \tilde{B}_2 is bounded when $\rho \rightarrow +\infty$ so $L(\beta^{1/2}, \tilde{B}_{2n}^{\alpha, \rho})$ is constant and $L(\beta^{1/2}, \tilde{B}^{\alpha, \rho}) = 0$ when ρ is near $+\infty$.

Assume that $L(\beta^{1/2}, \tilde{B}^{\alpha, \rho})$ is non constant then according to Theorem 3.1 the acceleration $\omega(\beta^{1/2}, \tilde{B}^{\alpha, \rho})$ is positive for $\rho < 1$ and negative for $\rho > 1$; there is thus a jump of -2 for $\omega(\beta^{1/2}, \tilde{B}^{\alpha, \rho})$. By definition of $\tilde{B}^{\alpha, \rho}$ we have

$$\begin{aligned} L(\beta^{1/2}, B^{\alpha, \rho}(y)) &= L(\beta^{1/2}, \tilde{B}^{\alpha, \rho}(y)) + \int_{\mathbb{S}_p^1} \ln \sqrt{\alpha - y^2} dy \\ &= L(\beta^{1/2}, \tilde{B}^{\alpha, \rho}(y)) + \max(0, \ln \rho) \end{aligned}$$

so there is a jump of -1 for $\omega(\beta^{1/2}, B^{\alpha, \rho})$ that is impossible by convexity of L (see §3).

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